THE OPEN UNIVERSITY

Mathematics: A Third Level Course

M332 Complex Analysis

M332
Assignment Booklet

Contents

			Cut-off date
2	TMA M332 01	(Units 0, 1 and 2)	10 April 1991
4	TMA M332 02	(Units 3, 4 and 5)	29 May 1991
6	TMA M332 03	(Units 6, 8 and 9)	17 July 1991
8	TMA M332 04	(Units 10, 11 and 13/14)	11 September 1991

Please send all your answers to each tutor-marked assignment (TMA), together with an appropriately completed assignment form (PT3), to reach your tutor on or before the appropriate cut-off date shown above.

You will find instructions on how to fill in the PT3 form in the current *Student Handbook*. Remember to fill in the correct *Assignment Number* as listed above and allow sufficient time in the post for each assignment to reach its destination on or before the cut-off date.

The marks allocated to each part of each TMA question are indicated in brackets in the margin.

Please show all your working unless explicitly asked not to. *All questions will be used for assessment purposes, except in TMA 04.*

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On this assignment *all* questions will be used for assessment purposes.

Question 1 (Unit 0)

Let the function f be defined by the formula

$$f(x) = 2x^2 + 3x + 5, \qquad x \text{ real.}$$

Prove that f is differentiable at the point 1, with derivative $f'(1) = 7$, by finding a formula for δ in terms of ε such that Definition 3 of *Unit 0*, Section 4.1, is satisfied for *each* given $\varepsilon > 0$. [5]

Question 2 (Unit 0)

Let the function f be defined by the formula

$$f(x) = \frac{2x^2 - 3x + 15}{(x-3)(x^2+3)}, \qquad x \text{ real}, x \neq 3.$$

(i) Using Theorem 32 of *Unit 0*, find an upper estimate for

$$\left| \int_{-2}^{2} f(x)\,dx \right|. \qquad [7]$$

(ii) Express $f(x)$ in partial fraction form, and hence determine the Taylor series for f at 0, stating explicitly the term in x^n, and the radius of convergence of the series. [13]

Question 3 (Unit 1)

Let $z_0 = \dfrac{-32}{1 - \sqrt{3}i}$.

(i) Find

(a) $\operatorname{Re} z_0$,

(b) $\operatorname{Im} z_0$,

(c) $|z_0|$,

(d) $\operatorname{Arg} z_0$. [7]

(ii) Express z_0 in polar form, and hence determine all the fourth roots of z_0, giving your answers in the form $a + ib$, where a and b are real numbers. [6]

(iii) Produce a *real* equation (i.e. one whose coefficients are real) which has z_0 as a root. [2]

Question 4 (Unit 1)

Use de Moivre's Theorem (*Unit 1*, page 30) and the Binomial Theorem to prove that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$$

where θ is a real number. [5]

Question 5 (Unit 1)

Draw diagrams showing the following sets of points in the complex plane. In each case indicate which points of the boundary belong to the set, and which points do not.

(i) $\{z : 2 \leq |z| \leq 3\} \cap \{z : 0 \leq \operatorname{Arg} z \leq \pi\}$ [4]

(ii) $\{z : \operatorname{Im} z < 2 \operatorname{Re} z + 6\}$ [4]

(iii) $\left\{z : \operatorname{Re}\left(\dfrac{z-1}{z+1}\right) \leq 0\right\}$ [7]

Question 6 (Unit 2)

Let the function f be defined by the formula

$$f(z) = \frac{z-1}{z+1}, \qquad z \neq -1.$$

(i) Show that f is one-one, and find the rule and domain of f^{-1}. [4]

(ii) Use your formula for f^{-1} to find the image under f of the disc $\{z : |z| < 1\}$, and sketch this image set. [6]

Question 7 (Unit 2)

(i) For this part you may use any theorems or results from *Unit 2* without proof, provided that you make clear which theorems or results you are using and that you check that the conditions under which the theorems are true hold here.

(a) Find the domain of the function f defined by

$$f(z) = \frac{z^2}{z^2 - 2z + 2}.$$ [4]

(b) Evaluate $\lim_{z \to 2i} f(z)$, and express your answer in the form $a + ib$, where a and b are real. [4]

(c) Prove that f is continuous on the set $\{z : |z| \leq 1\}$. [3]

(d) Prove that f is bounded on the set $\{z : |z| \leq 1\}$. [3]

(ii) Using only the definitions of limit and continuity, show that $g(z) = 3\overline{z} + 5 \operatorname{Re} z$ is continuous at 7. [6]

Question 8 (Unit 2)

Let S be the set $\{z : 1 \leq |z| < 2\} \cup \{0\} \cup \{2\}$.

(i) Write down (no explanation required):

(a) the set of interior points of S;

(b) the closure of S;

(c) the set of boundary points of S;

(d) the set of isolated points of S. [6]

(ii) For each of the sets considered in part (i) decide whether or not it is (A) a region, (B) compact. [4]

On this assignment *all* questions will be used for assessment purposes.

Question 1 (Unit 3)

Express the following complex numbers in the form $a + ib$, where a and b are real:

(i) $\exp(2 + \pi i/4)$; [3]

(ii) $1 + 2\sinh^2(\pi i/6)$; [5]

(iii) $\mathrm{Log}(-2i)$; [2]

(iv) all possible values of $(-2i)^{-1}$. [5]

(Simplify as far as possible but give answers in exact form, not as a decimal.)

Question 2 (Unit 3)

For each of the functions

(i) $f(z) = \cosh(z^3)$, [3]

(ii) $g(z) = \mathrm{Log}(-z)$, [4]

(iii) $h(z) = \sqrt{(1 - iz^2)}$, where $\sqrt{}$ denotes the principal branch, [8]

(a) find its domain, and sketch it;

(b) write down its derivative.

Question 3 (Unit 3)

Use the Cauchy–Riemann equations to show that function

$$f(z) = \mathrm{Im}\, z + i\,\mathrm{Re}\, z$$

is not differentiable at any point of $\mathbf{C}$. [5]

Question 4 (Unit 4)

(i) Sketch the paths of the arcs γ_1 and γ_2 defined by:

$$\gamma_1(t) = i + t^3(1 + i), \qquad t \in [-1, 1];$$
$$\gamma_2(t) = 3 + 4e^{4it}, \qquad t \in [0, \pi].$$ [4]

(ii) For each of the arcs γ_1 and γ_2 write down (no explanation required) whether or not it is

(a) simple, (b) closed, (c) simple-closed, (d) smooth. [6]

Question 5 (Unit 4)

(i) Evaluate $\int_{\gamma_1} \overline{z}\, dz$, where γ_1 is the parabolic arc defined by $\gamma_1(t) = t + it^2$, $t \in [0, 1]$. [6]

(ii) Evaluate $\int_{\gamma_2} \overline{z}\, dz$, where γ_2 is the line segment $[0, 1 + i]$. [6]

(iii) Use your answers to parts (i) and (ii) to deduce that the function $z \longrightarrow \overline{z}$ cannot be the derivative of any entire function f. [3]

Question 6 (Unit 4)

Let Γ be the square contour with vertices at $2-2i$, $2+2i$, $-2+2i$, $-2-2i$, and let I be defined by

$$I = \int_\Gamma \frac{\sin z}{z^2+1}\,dz.$$

Use the Estimation Theorem to obtain an upper bound for $|I|$. Note that you must check that the conditions of the theorem hold. [10]

Question 7 (Unit 5)

Evaluate, mentioning by name any important results that you use (and checking necessary conditions), the integrals:

(i) $\displaystyle\int_C \frac{z^2}{(z+2)^2(z-3)}\,dz$,

(a) when C is the circle $\{z : |z| = 1\}$, [3]

(b) when C is the circle $\{z : |z-3| = 2\}$, [4]

(c) when C is the circle $\{z : |z+3| = 3\}$. [5]

(ii) *Without doing the calculations involved*, write down a strategy which would allow you to evaluate the integral of part (i) when C is the circle $\{z : |z| = 4\}$. [3]

Question 8 (Unit 5)

(i) Prove, using Liouville's Theorem or otherwise, that for any real number $k > 0$ there is a $z_0 \in \mathbf{C}$ for which $|\cos z_0| > k$. [4]

(ii) Write down

(a) a function with period 1;

(b) a function with period i. [3]

(iii) In this part you are required to establish the proposition that there are no non-constant entire functions which are doubly periodic with both a real period and an imaginary period.

Suppose that f is entire, that a and b are real and positive, and that f satisfies

$$f(z+a) = f(z), \quad f(z+bi) = f(z) \quad \text{for all } z \in \mathbf{C}.$$

(a) Explain why f is bounded on the rectangle

$$S = \{z : 0 \le \operatorname{Re} z \le a \text{ and } 0 \le \operatorname{Im} z \le b\}.$$

(b) Deduce that f is a constant function and that the above proposition is established. [8]

On this assignment *all* questions will be used for assessment purposes.

Question 1 (Unit 6)

Let f be the function defined by $f(z) = \dfrac{z}{z+3}$.

(i) Write down the set of points on which f is analytic. [1]

(ii) Determine the Taylor series for f:

(a) at the point 0; [5]

(b) at the point 1. [7]

(In each case you should include a general term of the series.)

(iii) Write down the radius of convergence of each of the two series in part (ii). [2]

Question 2 (Unit 6)

(i) Using the methods of Section 6 of *Unit 6*, or otherwise, find the first three non-vanishing terms in the Taylor series for the following functions about 0:

(a) $z \longrightarrow z(e^z - 1)^2$; [4]

(b) $z \longrightarrow \operatorname{Log} \cos z$; [6]

(c) $z \longrightarrow \tan z$. [2]

(ii) Write down the order of the zero of each of the functions in part (i) at the point 0. [3]

Question 3 (Unit 6)

An entire function f satisfies the equation $f(1/n) = 1/n$ for every positive integer n. Prove that $f(z) = z$ for all z in $\mathbf{C}$. [5]

Question 4 (Unit 8)

For each of the following functions, write down:

(a) its singularities;

(b) whether these singularities are removable singularities, essential singularities, or poles;

(c) the order of any poles.

(i) $z \longrightarrow z \sinh\left(\dfrac{1}{z}\right)$

(ii) $z \longrightarrow \dfrac{z-1}{(z^2-1)(z+1)}$

(iii) $z \longrightarrow \dfrac{1}{e^z - 1}$ [15]

Question 5 (Unit 8)

Let f be the function defined by $f(z) = \dfrac{4}{z(z-2)}$.

(i) Find a series expansion for f about 0, convergent on the punctured disc $\{z : 0 < |z| < 2\}$. [4]

(ii) Find a series expansion for f about 0, convergent on the region $\{z : |z| > 2\}$. [4]

(iii) Find a series expansion for f about 1, convergent on the disc $\{z : |z-1| > 1\}$. [6]

(iv) Write down the value of $\text{Res}(f, 0)$. [1]

(In parts (i), (ii) and (iii) you should give a general term of the expansion.)

Question 6 (Unit 9)

Find the maximum value of the function

$$z \longrightarrow |\exp(1 + iz^2)|$$

on the disc $\{z : |z| \leq 2\}$.

List *all* the points where this maximum is attained. [10]

Question 7 (Unit 9)

For each of the following sets state (a) whether or not it is connected, and (b) whether or not it is simply-connected.

(i) $\mathbf{C} - \{0\}$

(ii) $\mathbf{C} - \{z : |z-3| = 1\}$

(iii) $\mathbf{C} - \{z : z \text{ is not zero nor a negative real number}\}$ [5]

Question 8 (Unit 9)

Let f be a function non-constant and analytic on $\{z : |z| < 1\}$ and continuous on $\{z : |z| \leq 1\}$ whose modulus is constant on $\{z : |z| = 1\}$. Use the Maximum and Minimum Principles to prove that f has at least one zero in $\{z : |z| < 1\}$. [10]

Question 9 (Unit 9)

Find, using the Residue Theorem or any other results you wish, the possible exact values of

$$\int_\Gamma \frac{ze^z}{z^2+1}\,dz,$$

where Γ is a simple-closed contour which does not pass through $\pm i$ and is traversed in the positive direction. [10]

In this assignment you should answer *all* of Questions 1–7, which are on *Units 10* and *11*, plus *either* Question 8 on *Unit 13 or* Question 9 on *Unit 14*.

Question 1 (Unit 10)

For each of the following functions f and points α, determine the residue of f at α.

(i) $f(z) = \dfrac{z}{\sin z}$, $\quad \alpha = 0.$

(ii) $f(z) = \dfrac{z}{\sin z}$, $\quad \alpha = \pi.$

(iii) $f(z) = \dfrac{1}{\sin\left(\frac{1}{z}\right)}$, $\quad \alpha = 0.$ [5]

Question 2 (Unit 10)

Find a function f such that

$$\int_0^{2\pi} \frac{1}{17 - 8\cos\theta} d\theta = \int_C f(z)dz,$$

where $C = \{z : |z| = 1\}$, and hence evaluate the first integral. [10]

Question 3 (Unit 10)

Evaluate, giving your answer in an exact form,

$$\int_{-\infty}^{\infty} \frac{t \sin 5t}{t^4 - 1} dt.$$

Quote any theorem that you use and check that the conditions of the theorem hold. [10]

Question 4 (Unit 10)

(i) Locate the poles of the function f defined by

$$f(z) = \frac{\pi}{z^3 \cos(\pi z)},$$

and calculate the residues of f at these poles. [8]

(ii) By considering $\int_{S_N} f$, where f is as defined in part (i), and S_N is the square contour with vertices $N(1+i)$, $N(-1+i)$, $N(-1-i)$ and $N(1-i)$, N a natural number, prove that

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}.$$

(You may assume that there is a positive constant K independent of N such that $|\sec(\pi z)| \leq K$ for all z on S_N.) [7]

Question 5 (Unit 11)

(i) Use Theorem 3 of *Unit 11* to prove that

$$f(z) = \sum_{n=1}^{\infty} e^{-nz}$$

defines a function which is analytic on $R = \{z : \mathrm{Re}\, z > 0\}$. [8]

(ii) Write down a function g which is analytic on $S = \{z : 0 < \mathrm{Im}\, z < 2\pi\}$ such that (f, R) and (g, S) are direct analytic continuations of each other. [2]

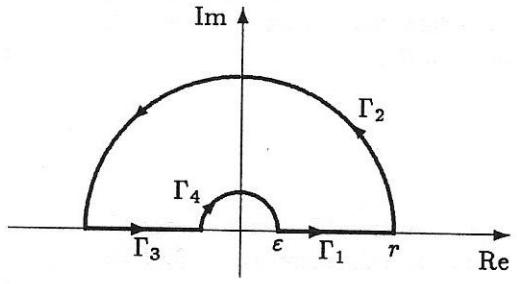
Question 6 (Unit 11)

By considering the integral of the complete analytic function

$$f : z \longrightarrow \frac{\log z}{1+z^4}$$

around the contour Γ shown in the diagram, evaluate the real improper integral

$$\int_0^\infty \frac{\log x}{1+x^4}dx.$$



You may assume that the integrals round the semi-circular paths tend to 0 as $r \to \infty$ and $\epsilon \to 0$. [10]

Question 7 (Unit 11)

Use Rouché's Theorem to determine the number of zeros of the polynomial $z^5 - 5z + 7$ which lie in the annulus $\{z : 1 < |z| < 2\}$.

Explain why this polynomial cannot have a purely imaginary zero. [10]

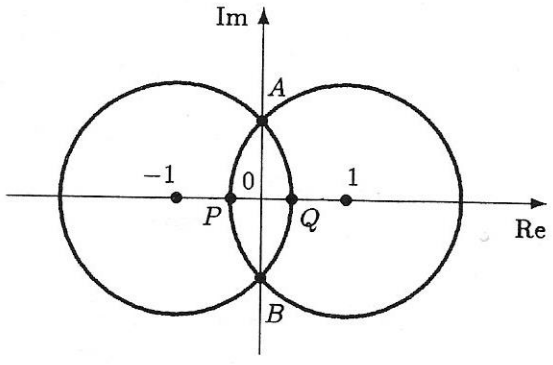
Question 8 (Unit 13)

(i) Let D be the intersection of the discs $\{z : |z-1| \leq \sqrt{2}\}$ and $\{z : |z+1| \leq \sqrt{2}\}$ (see diagram).

(a) Find a bilinear transformation f which maps i to 0, 0 to 1 and $-i$ to ∞.

(b) Find the images under f of the circular arcs APB, AQB. At what angle do these images intersect?

(c) Use the result of part (b) to find an analytic equivalence between the interior of D and the upper half-plane. [20]



(ii) Find an analytic equivalence of $\{z : \pi/3 < \operatorname{Arg} z < 2\pi/3\}$ with $\{z : |z| < 1\}$. [4]

(iii) Find an analytic equivalence of $\{z : |z-i| < 3\}$ with $\{z : \operatorname{Re} z < 1\}$. [6]

(You may quote without proof any results which appear in *Unit 13* and use diagrams to illustrate your arguments throughout this question.)

Question 9 (Unit 14)

(i) (a) Show that $\mathcal{L}[\sinh](p) = \dfrac{1}{p^2 - 1}$. [2]

(b) Suppose that f satisfies the differential equation

$$f''(t) + f'(t) - 2f(t) = \sinh t,$$

where $f(0) = 0$ and $f'(0) = 1$. Determine the Laplace transform F of f. [5]

(c) Show that the function F obtained in part (b) satisfies the conditions of Theorem 1 (page 23 of *Unit 14*), and hence find, using a residue method, the solution $f(t)$ of the above differential equation. [14]

(ii) Let f be the function defined by

$$f(t) = \frac{1}{1 + t^2},$$

and let F be its Laplace transform. Use Watson's Lemma (Theorem 7, page 47 of *Unit 14* — and note the Stop Press correction to this theorem) to find an asymptotic expansion for $F(p)$, where p is real and large. [9]

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